A STUDY OF ARITHMETICALLY SYMMETRICAL BANDPASS FILTERS

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INTRODUCTION

In the transmission of modulated waves and in high speed data transmission systems, it is often desirable to have bandpass filters with arithmetically symmetrical attenuation and group-delay characteristics.

The possibility of obtaining such a symmetry is practically always assured when the passband is narrow in comparison with the center frequency. On the other hand, it is lost when the bandwith is an appreciable fraction of the center frequency. It is well known that the conventional low-pass to bandpass filters transformation provides a bandpass filter having geometrical symmetry in attenuation characteristic and exhibiting no symmetry at all in the group-delay characteristic.

Problems of designing lumped element bandpass filters having arithmetically symmetrical characteristics in the case of large bandwith are investigated in this paper.

Two approaches are considered: the characteristic function approach and Butterworth criterion approach.

The characteristic function approach deals with periodic bandpass filter characteristics which have exact arithmetic symmetry; that is, the filter must have an infinite number of pass bands and is essentially a transmission-line filter. Use of lumped circuit elements induces two approximations. One of these is approximation of the ideal filter characteristic; the other is approximation of arithmetic symmetry. Arithmetic symmetry

inside the stop bands is of no consequence, hence, its approximation can be restricted to pass and transition bands. A method is presented to perform these approximations. This method is partly graphical and numerical.

In the Butterworth criterion approach, the bandpass transfer function without zeros other than at zero and infinite frequencies are investigated in order to obtain the maximum possible arithmetic symmetry of attenuation and group-delay of any degree Butterworth pass band behavior.

Design procedures according to Butterworth criteron are considered in this report.

Both approaches are physically realizable by reactive ladder two-port networks terminated in a resistor.

PERIODIC FREQUENCY TRANSFORMATION

The desirable filter spectrum has exact arithmetic symmetry with respect to a frequency $\omega_o \neq 0$, and it exhibits the same type of symmetry with respect to the frequency $\omega = 0$. Since this function has a repeated series of inversions with respect to these frequencies it must be periodic with period $2\omega_o$. These structures are characterized in terms of circular or hyperbolic, instead of rational, functions and hence are transmission line filters.

Theorem: Only open-circuited and short-circuited lossless LC transmission line segments, or their equivalents, may be used in a filter to obtain a spectrum having arithmetic symmetry with respect to a frequency $\omega_o \neq 0$, and the spectrum will be periodic.

Proof: If the specified transmission line segments and resistor terminations are employed, the filter's transfer function will be a rational function of the delay operator e^{-p}. If p is replaced by $j\omega$ then arithmetic symmetry and periodicity are evident.

The most general filter of this type can be obtained from the conventional low-pass filter function with pass band $-1 \leq \Omega \leq 1$ by the frequency transformation

$$\Omega = -\frac{h}{\tan \frac{\pi \omega}{2 \omega_o}} \tag{1}$$

or

$$P = j\Omega = \frac{h}{j\tan \frac{\pi\omega}{2\omega_o}} = \frac{h}{\tanh \frac{\pi\omega}{2\omega_o}} \tag{1'}$$

where the parameter "h" is related to the band edges of the transformed filter:

$$h = \tan \frac{\pi\omega_l}{2\omega_o} = -\tan \frac{\pi\omega_u}{2\omega_o} \tag{2}$$

where ω_l and ω_u are the lower and upper edges of the first pass band respectively.

CHARACTERISTIC - FUNCTION APPROACH

I. Transformation of the Characteristic Function

Let the transfer function of a two port network be:

$$e^{a+jb} = T(p) = \frac{g(p)}{f(p)} \tag{3}$$

where

a is the transducer loss (or attenuation)

b is the phase angle

$p = j\,\omega$

g(p) is a Hurwitz polynomial

f(p) is an even or odd polynomial with imaginary zeros and of degree not greater than that of g(p).

Also,

$$e^{2a} = T(p)T(-p) = \frac{g(p)g(-p)}{(f(p))^2} = 1 + \phi(p)\phi(-p) \tag{4}$$

where

$$\phi(p) = \frac{h(p)}{f(p)} \tag{5}$$

is the characteristic function, h(p) is a real arbitrary Hurwitz polynomial relatively prime to f(p).

Let us consider a normalized low-pass filter with pass band

$0 \leq \Omega \leq 1$,

and stop band

$$1 \leq k \leq \Omega \leq \infty$$

Its attenuation characteristic is shown in Fig. 1. Now we apply the periodic frequency transformation (1). This transformation yields a filter with an infinite number of pass bands each arithmetically symmetrical and centered at

$$(2k+1)\,\omega_o \qquad k = 0, 1, 2, 3, \cdots.$$

A lumped element realization could be obtained for this transformation (1) by approximating the hyperbolic tangent function with a rational function. There are several approximations for this hyperbolic function which retain the rational form; namely,

1. Partial fraction expansion of tanhx;
2. Partial fraction expansion of coth x;
3. Continued fraction expansion of these functions.

The disadvantages of these methods are that one or more periodic pass bands are generated and that the number of lumped elements required is excessive. The problem is to modify extraneous pass bands.

II. Truncation of Characteristic Function

Let us look at the attenuation characteristic of the transformed band-pass filter in Fig. 2. There exist zeros and poles in the region

$$0 \leq \omega \leq 2\omega$$

and this constellation repeats infinitely many times in both directions. We wish to retain the singularities in the fundamental region

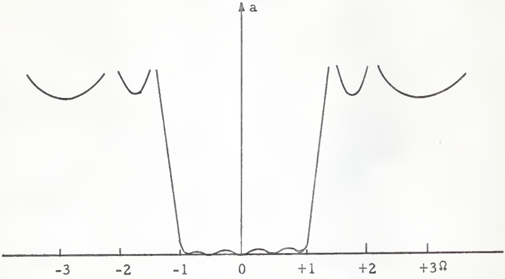

Fig. 1 Normalized low-pass characteristic.

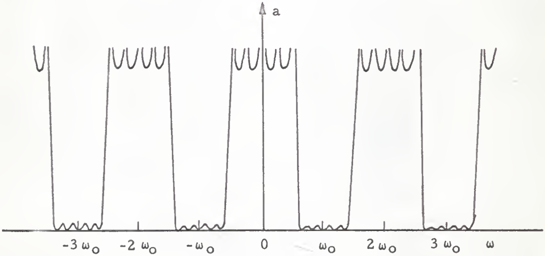

Fig. 2 Periodic bandpass characteristic.

$$-2\omega_o \leq \omega \leq 2\omega_o$$

and discard all others. We recognize that the characteristic function $\Phi(P)$ of the low-pass filter directly exhibits these singularities. We can write

$$\Phi(P) = \Phi(h \operatorname{cotanh} \frac{\pi p}{2\omega_o}) = \phi_t(p). \tag{6}$$

If we combine all singularities of $\phi_t(p)$ in the fundamental region into the function

$$\phi^*(p),$$

it is observed that half of the poles at Im $p = \pm 2\omega_o$ must be discarded. Consequently $\phi^*(p)$ need not be rational. Considering the poles of the function $\phi_t(p)$ we have

$$\tanh \frac{\pi p}{2\omega_o} = 0$$

or

$$\tanh \frac{\pi p}{2\omega_o} = \tanh \left(\frac{\pi p}{2\omega_o} + 2\pi kj \right) = 0. \tag{7}$$

$$\frac{\pi p}{2\omega_o} + 2\pi kj = 0$$

$$p + 4kj\omega_o = 0 \tag{8}$$

Similarly, we consider the zeros of the function $\phi_t(p)$ and obtain

$$\tanh \frac{\pi p}{2\omega_o} = 0$$

as a result this reduces to equation (8). Therefore, $\phi_t(p)$ can be written

as (apart from a constant multiplier)

$$\phi_t(p) = \prod_{k=-\infty}^{+\infty} \phi^*(\, p+4kj\,\omega_o\,). \tag{9}$$

The initial approximation will be $\phi^*(p)$ itself.

Discarding all singularities outside of the fundamental range results in loss of arithmetic symmetry of the bandpass characteristic. As a result, some correction is needed. As a measure of the deviation from the ideal case we select the function:

$$D = + \log \left| \frac{\phi^*(p)}{\phi_t(p)} \right|_{p=j\omega} \tag{10}$$

which is a smooth function in the fundamental region and can be evaluated relatively easily.

The final step is to select a function n(p) with the following properties:

(1) The function

$$\phi(p) = \phi^*(p)n(p) = \frac{h(p)}{f(p)}$$

is a rational realizable characteristic function.

(2) n(p) does not have pure imaginary zeros. This is necessary in order to avoid spurious pass bands.

(3) The function

$$D + \log n(p)$$

is approximately constant inside the fundamental region or the

restricted region consisting of the pass and transition bands. This last step is best performed numerically or graphically.

In order to expedite this last step, let us take another look at the deviation function, equation (10). This function is a measure of the deviation of the first approximation from the ideal, symmetrical case and therefore is an indirect measure of the asymmetry itself. It is a weighted measure such that the deviation is weighted considerably heavier for low losses (pass band) than for high losses where the weighting is essentially unity. This weighting is quite convenient since the symmetry is only meaningful in the pass and transition bands, that is, in the low and intermediate loss regions.

Let us consider a factor of the form

$$(p^2 + \Omega_i^2)^{\pm 1} = (\Omega_i^2 - \Omega^2)^{\pm 1}$$

in the low-pass function $\Phi(p)$. The corresponding factor in $\phi_t(p)$ will be of the form

$$\left(h^2 \left(\cotan^2\left(\frac{\pi\omega}{2\,\omega_o}\right) - \cotan^2\left(\frac{\pi\omega_i}{2\,\omega_o}\right) \right) \right)^{\pm 1}$$

with

$$h \cotan \frac{\pi\omega_i}{2\,\omega_o} = \Omega_i.$$

We write the corresponding factor in $\phi^*(p)$ in the form:

$$\left(\frac{(\omega^2 - \omega_i^2)(\omega^2 - (2\omega_o - \omega_i)^2)}{\omega^2} \right)^{\pm 1} .$$

This will lead to a term in the deviation function which is given by (apart from an uninteresting additive constant):

$$D_i = \pm \log \left| \left(\frac{2\omega_o}{\pi\omega} \right)^2 \frac{\left(1 - \frac{\omega^2}{\omega_i^2} \right) \left(1 - \frac{\omega^2}{(2\omega_o - \omega_i)^2} \right)}{\left(\operatorname{cosec}^2 \frac{\pi\omega}{2\omega_o} - \operatorname{cosec}^2 \frac{\pi\omega_i}{2\omega_o} \right)} \right| . \tag{11}$$

The limiting case $\Omega_i = 0$ corresponds to a double at midband with:

$$D_o = 2 \log \left| \frac{\left(\tan \frac{\pi\omega}{2\omega_o} \right) \left(1 - \frac{\omega^2}{\omega_o^2} \right)}{\frac{\pi\omega}{2\omega_o}} \right| . \tag{12}$$

The other limiting case of $\Omega_i \to \infty$ is clearly a double pole at $\omega = 0$ and single poles at $\pm 2\,\omega_o$ giving

$$D_\infty = -\log \left[1 - \left(\frac{\omega}{2\omega_o} \right)^2 \right]. \tag{13}$$

This is a one parameter set of curves whose two limiting cases are shown in Fig. 3 as a function of ω / ω_o. For more accurate calculation, it is better to use the modified function

$$D'_i = D_i - D_o \qquad\qquad i = 0, 1, 2, 3, \cdots$$

a few of which are shown in Fig. 4 with ω/ω_o as parameter. The value of D_i is in decades.

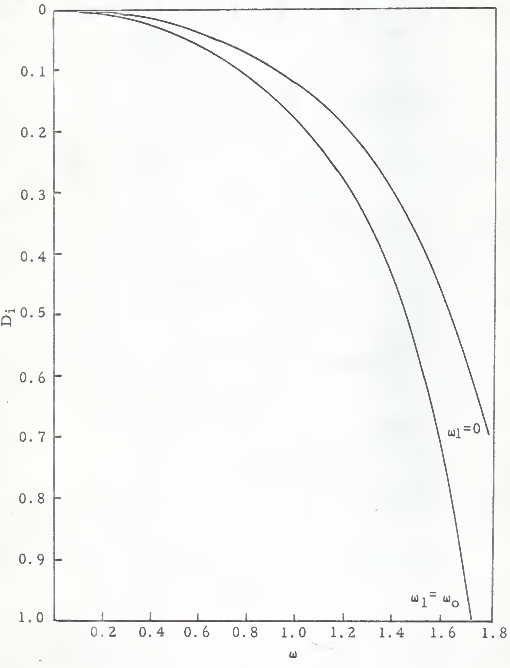

Fig. 3 Limiting cases of the deviation function

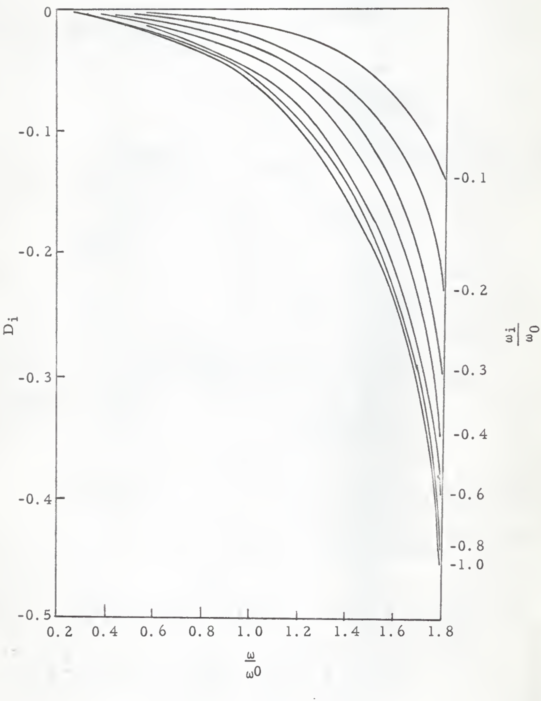

Fig. 4 Relative deviation functions

CONSTANT GROUP-DELAY BANDPASS FILTER

Group delay is defined to be:

$$\tau(\omega) = \frac{d\beta(\omega)}{d\omega}$$

where $\beta(\omega)$ is the phase of transfer function $T(p)$, $p=j\omega$.

Let us apply the periodic frequency transformation method to some low pass filter to obtain the bandpass filter and investigate the filter's group delay

$$\frac{d\beta_b}{d\omega} = \frac{d\beta_l}{d\Omega} \cdot \frac{d\Omega}{d\omega} \tag{14}$$

where

$$\frac{d\Omega}{d\omega} = h \left. \frac{dR(p)}{dp} \right|_{p=j\omega} .$$

Here β_b and β_l are the phases of the low pass and bandpass filters respectively and $R(p)$ is the reactive rational approximant to $\coth(\pi p/2)$ or approximately:

$$\frac{dR}{dp} \doteq \frac{d}{dp} \coth\frac{\pi p}{2}$$

$$= -\frac{\pi}{2} \frac{1}{\sinh^2 \frac{\pi p}{2}}$$

$$= \frac{\pi}{2} \left(1-\coth^2 \frac{\pi p}{2} \right)$$

$$= \frac{\pi}{2} \left(1 - \left(\frac{p}{h}\right)^2 \right) .$$

Consequently one can obtain

$$\frac{d\beta_b}{d\omega} \doteq \frac{h\pi}{2\omega_o}\left(1 + \left(\frac{\Omega}{h}\right)^2\right) \frac{d\beta_1}{d\Omega} \quad . \tag{15}$$

Hence, if $d\,\beta_1/d\Omega$ is constant, the transformed group delay $d\,\beta_b/d\omega$ is not constant. We are interested in starting with a low-pass filter having $d\,\beta_1/d\Omega$ not a constant and end with a band pass filter having constant group delay

$$\tau_b(\omega) \doteq \tau_o \,.$$

Let us start with a low-pass filter group delay of the form:

$$\frac{d\beta_1(\Omega)}{d\Omega} \doteq \frac{k}{h\left(1 + (\Omega/h)^2\right)} \tag{16}$$

where

$\beta_1(\Omega)$ is the phase of the normalized low-pass characteristic

and

$$k = \frac{2\,\tau_o\,\omega_o}{\pi} \quad .$$

Integrating this expression once, we have

$$\beta_1(\Omega) \doteq k \tan^{-1}\frac{\Omega}{h} \,. \tag{17}$$

In a slightly different form, Equation (17) becomes:

$$j\,\beta_1 \doteq k \tanh^{-1}\frac{j\Omega}{h} = k \tanh^{-1}\frac{P}{h} \,. \tag{18}$$

Furthermore, we know that tanh $j\,\beta$ must be the ratio of an odd and even

polynomial in P; that is

$$\tanh j\beta_1 = \frac{0(P)}{E(P)} \doteq \tanh \left(k \tanh^{-1} \frac{P}{h} \right) \tag{19}$$

One can obtain an approximation of the maximally flat type and the rational form by expanding the right side of Equation (19) into a continued fraction:

$$\tanh \left(k \tanh^{-1} \frac{P}{h} \right) = \cfrac{\frac{kP}{h}}{1 + \cfrac{(k^2-1)\frac{P^2}{h^2}}{3 + \cfrac{(k^2-4)\frac{P^2}{h^2}}{5 + \cfrac{(k^2-9)\frac{P^2}{h^2}}{7 + \cdots}}}} \tag{20}$$

Terminating the expansion with the Nth term, we obtain the rational approximant $O_N(P)/E_N(P)$ and the polynomial of degree N:

$$Q_N(P) = O_N(P) + E_N(P) \tag{21}$$

The polynomial will have Hurwitz character, if the included continued fraction coefficients are all positive, that is, if

$$k = \frac{2\, \tau_0 \omega_o}{\pi} \geq N - 1 \tag{22}$$

In case of equality, the continued fraction expansion terminates, and the approximation becomes an exact equality. Equation (22) becomes

$$k = \frac{2\tau_o \omega_o}{\pi} = N-1 = n$$

and we find that

$$\frac{O_N(P)}{E_N(P)} = \tanh\left(n \tan^{-1} \frac{P}{n}\right)$$

$$= \frac{(1 + \frac{P}{h})^n - (1 - \frac{P}{h})^n}{(1 + \frac{P}{h})^n + (1 - \frac{P}{h})^n} \tag{23}$$

and

$$Q_n(P) = (1 + \frac{P}{h})^n \tag{24}$$

This leads to the following:

<u>Theorem</u>: If a reference low-pass filter with transfer function

$$T_L(P) = \frac{g(P)}{f(P)} = \frac{(1 + \frac{P}{h})^n}{f(P)}, \tag{25}$$

where $f(P)$ is an even polynomial of degree less than n, is transformed according to

$$P = h \coth \frac{\pi p}{2\omega_o}$$

into a bandpass transfer function with exactly linear phase

$$T(P) = \frac{(1 + \coth \frac{\pi p}{2\omega_o})^n}{f(h \coth \frac{\pi p}{2\omega_o})} = e^{a + j\beta} \tag{26}$$

then the constant group-delay at all frequencies is

$$\tau_b(\omega) = \tau_o \frac{n\pi}{2\,\omega_o} . \tag{27}$$

From foregoing considerations on the design of a constant group-delay band-pass filter we can employ the following procedures: (a) select a low-pass function of the form in Equation (25); (b) determine characteristic function $\Phi(P)$; (c) apply the technique for attenuation characteristic modification.

BUTTERWORTH CRITERION APPROACH

I. Symmetry Conditions For the Attenuation Function

Consider a bandpass transfer function without zeros other than at zero and infinite frequencies. Let the transfer function be of the type:

$$T(p) = \frac{p^n}{g^m(p)}, \tag{28}$$

where $g^m(p)$ is a polynomial of degree m, and $p = j\omega$, ω is a real frequency. We see that T(p) has zeros at p=0 and $p=\infty$, and that $1 \leq n \leq m - 1$. This transfer function has a zero of degree n at zero frequency and a zero of degree m-n at infinite frequency.

The square of the modulus of T(p), at real frequencies is:

$$\left|T(p)\right|^2 = \frac{p^{2n}}{Q^m(p^2)}, \tag{29}$$

where $Q^m(p^2)$ is a polynomial of degree m in p^2. If $\left|T(p)\right|_o$ is the maximum value assumed by $\left|T(p)\right|$ at real frequencies, a relative transfer function can be defined as

$$T(p)_r = \frac{T(p)}{\left|T(p)\right|_o}, \tag{30}$$

whose modulus is given by the expression:

$$\left|T(p)_r\right|^2 = \frac{T(p)^{\ 2}}{\left|T(p)\right|_o^2}$$

$$= \frac{p^{2n}}{Q^m(p^2) \left|T(p)\right|_o^2} \tag{31}$$

The expression for the relative attenuation a_r is:

$$e^{2a_r} = \frac{1}{\left|T(p)\right|^2}$$

$$= \frac{Q^m(p^2) \left|T(p)\right|_o^2}{p^{2n}}$$

$$= 1 + H \frac{S^m(p^2)}{p^{2n}}. \tag{32}$$

In order to obtain maximum uniformity of attenuation in the pass-band as many zeros as possible should occur at real frequencies. It follows from Equation (32) that

$$e^{2a_r} = 1+H \frac{(\omega^2+\omega_j^2)(\omega^2-\omega_1^2)^2 \ldots (\omega^2-\omega_{(m-1)/2}^2)^2}{\omega^{2n}} \tag{33}$$

when m is odd, and that

$$e^{2a_r} = 1+H \frac{(\omega^2-\omega_1^2)^2 (\omega^2-\omega_2^2)^2 \ldots (\omega^2-\omega_{m/2}^2)^2}{\omega^{2n}} \tag{34}$$

when m is even. In the Equations (33) and (34), ω_j, ω_1, ω_2, ... are

positive real. If the zeros of $S^m(p)$ in Equation (32) occur with even order multiplicity, then a_r cannot be negative. For each value of m and n, filter design will require the determination of H and of ω_j, ω_1, ω_2, ... which lead to the most satisfactory attenuation characteristics.

Regarding the choice of the values of ω_j, ω_1, ω_2, ... two fundamental criteria, Chebyshev criterion in Fig. 5, and Butterworth criterion in Fig. 6. are most frequently employed. We will restrict attention to the Butterworth criterion. This criterion keeps ω_1, ω_2, ... coincident in a single angular frequency ω_o, at which frequency, the maximum number of derivatives of a_r are set equal to zero. In this case, equations (33) and (34) become:

$$e^{2a_r} = 1 + H \frac{(\omega^2+\omega_j^2)(\omega^2-\omega_o^2)^{m-1}}{\omega^{2n}} ; \tag{35}$$

and

$$e^{2a_r} = 1 + H \frac{(\omega^2-\omega_o^2)^m}{\omega^{2n}} . \tag{36}$$

Formulas (35) and (36) will be the basis for approximating arithmetic symmetry conditions.

In the case of m even, we rewrite Equation (36) in the form:

$$e^{2a_r} = 1 + H(\omega-\omega_o)^m \cdot f_2(\omega) \tag{37}$$

where

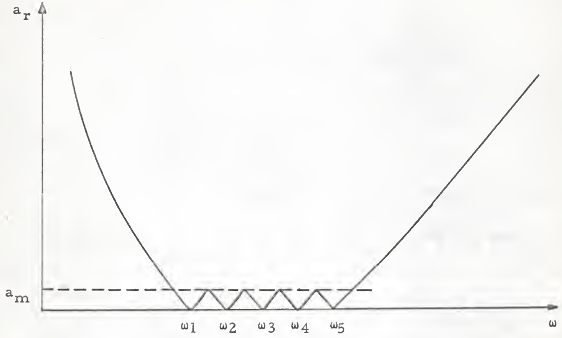

Fig. 5 Chebyshev attenuation Characteristic.

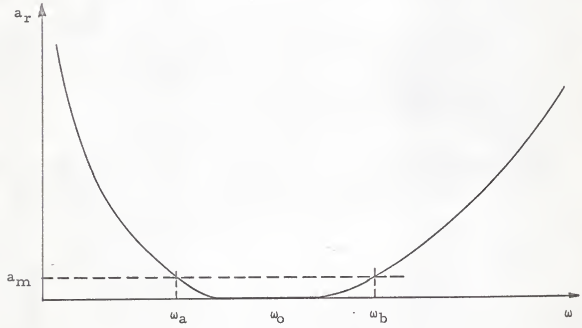

Fig. 6 Butterworth attenuation Characteristic.

$$f_2(\omega) = \frac{(\omega + \omega_o)^m}{\omega^{2n}} . \tag{38}$$

It is clear from expression (37) that the part $H(\omega - \omega_o)^m$ has even symmetry, therefore, in order that the function e^{2a_r} has even symmetry, then either $f_2(\omega)$ must be a constant, or it must have even symmetry relative to ω_o, at least in a fairly large interval including the bandpass region. It is observed that for narrow relative pass-band, ω very near ω_o, $f_2(\omega)$ can be considered as a constant with a value of

$$f_2(\omega) = 2^m \omega_o^{m-2n} \tag{39}$$

Let the function $f_2(\omega)$ be expanded in power series about ω_o:

$$f_2(\omega) = 2^m \omega_o^{m-2n} \left(1 + a_1 \frac{\omega - \omega_o}{\omega_o} + a_2 \left(\frac{\omega - \omega_o}{\omega_o}\right)^2 \right.$$

$$\left. + a_3 \left(\frac{\omega - \omega_o}{\omega_o}\right)^3 + \cdots \right) \tag{40}$$

wherein

$$a_1 = \frac{1}{2}(m-4n)$$

$$a_2 = \frac{1}{2}\left(2n(2n+1) - 2nm + \frac{m(m-1)}{4}\right)$$

$$a_3 = \frac{1}{2}\left(\frac{m(m-1)(m-2)}{24} - \frac{nm(m-1)}{2}\right.$$

$$\left. + mn(2n+1) - \frac{2n(2n+1)(2n-2)}{3}\right) . \tag{40}$$

From Equation (40) we observe that for narrow relative band width, $\omega - \omega_o$, $f_2(\omega)$ is equal to $2^m \omega_o^{m-2n}$, a constant. For large relative band with, the expression (40), taking into account the values of the coefficients, a_1, a_2, a_3, suggests choice of

$$m = 4n \tag{41}$$

for obtaining the maximum possible symmetry. Introducing this condition into Equation (40) yields

$$f_2(\omega) = 2^m \omega_o^{m/2}\Big(1 + \frac{m}{8}\Big(\frac{\omega - \omega_o}{\omega_o}\Big)^2 + - \frac{m}{8}\Big(\frac{\omega - \omega_o}{\omega_o}\Big)^3 + \ldots\Big). \tag{41}$$

In the case of odd m, we proceed in a similar manner. Equation (35) can be written as

$$e^{2a_r} = 1 + H(\omega - \omega_o)^{m-1} \cdot f_1(\omega), \tag{42}$$

wherein

$$f_1(\omega) = \frac{(\omega^2 + \omega_j^2)\ (\omega + \omega_o)^{m-1}}{\omega^{2n}} . \tag{43}$$

This time the part $H(\omega - \omega_o)^{m-1}$ has even symmetry and therefore $f_1(\omega)$ must be a constant or have an even symmetry relative to ω_o in order that the function e^{2a_r} is an even function in an interval including the passband.

As before, expanding $f_1(\omega)$ into power series about ω_o yields the following expression:

$$f_1(\omega) = 2^{m-1}\,\omega_o^{\,m-1-2n}(\omega_o^2+\omega_j^2)\left(1 + b_1\,\frac{\omega-\omega_o}{\omega_o} + b_2\left(\frac{\omega-\omega_o}{\omega_o}\right)^2 + b_3\left(\frac{\omega-\omega_o}{\omega_o}\right)^3 + \cdots\right), \qquad (44)$$

where

$$b_1 = \frac{1}{2}\left(m-1-4n+\frac{4}{1+\dfrac{\omega_j^2}{\omega_o^2}}\right),$$

$$b_2 = \frac{1}{2}\left(2n(2n+1) + \frac{(m-1)(m-2)}{4} - 2n(m-1) + 2\,\frac{m-4n}{1+\dfrac{\omega_j^2}{\omega_o^2}}\right),$$

$$b_3 = \frac{1}{2}\left(\frac{(m-1)(m-2)(m-3)}{24} - \frac{2n(2n+1)(2n+2)}{3} + n(2n+1)(m-1) - \frac{n(m-1)(m-2)}{2} + \frac{1}{1+\dfrac{\omega_j^2}{\omega_o^2}}\left(\frac{(m-1)(m-2)}{2} + 4n(2n+1)-4n+m-1-4n(m-1)\right)\right) \qquad (45)$$

Setting, as before, $b_1 = 0$, we observed that m-1-4n is an even number, and therefore

$$\frac{4}{1 + (\omega_j / \omega_a)^2}$$

must be even. For all possible finite values of ω_j, the above form ranges between 4 when $\omega_j = 0$ and 2 when $\omega_j = \omega_o$. Between these solutions, the first corresponds to lowering by one unit the degree of the denominator in Equation (28), falling back to the case of m even, as previously examined. Adopting the second solution, we have

$$\begin{aligned} \omega_j &= \omega_o \\ m + 1 &= 2n \end{aligned} \tag{46}$$

Equation (44) becomes:

$$f_1(\omega) = 2^m \omega_o^{\frac{m+1}{2}} \left(1 + \frac{m+3}{8} \left(\frac{\omega - \omega_o}{\omega_o}\right)^2 + \right.$$

$$\left. - \frac{m+3}{8} \left(\frac{\omega - \omega_o}{\omega_o}\right)^3 + \cdots \right) . \tag{47}$$

A summary of what we have developed now follows:

1. The ideal characteristic with maximum attenuation flatness having arithmetical symmetry, is:

$$e^{2a_r} = 1 + M(\omega - \omega_o)^{2k} \tag{48}$$

where k is any integer and M is a constant H multiplying the constant term of $f_1(\omega)$ or $f_2(\omega)$.

2. The Physically realizable characteristics are of the type:

$$e^{2a_r} = 1 + H(\omega - \omega_o)^{2k} \cdot f(\omega) \qquad (49)$$

3. If in Equation (49) k is even, the most favorable $f(\omega)$, as regards symmetry, is given by Equation (41), on the basis of m=2k, and n=k/2.

4. If in Equation (49) k is odd, the most favorable $f(\omega)$, regards symmetry, is given by (47), on the basis of m = 2k + 1, n=(k+1)/2, and $\omega_j = \omega_o$.

By introducing into Equation (28) the specified values of m and n, and because of considerations of 1 and 4, it is possible to write down the expression of the most symmetrical transfer function corresponding to an attenuation function of the prescribed order k. Table 1 shows the required formulas for values of k between 1 and 6.

II. Group-delay Characteristic

The group-delay of the transfer function of the type in Equation (28) is:

$$\tau = j\left(\frac{d\beta_1}{dp} + \frac{d\beta_1}{dp} + \frac{d\beta_2}{dp} + \frac{d\beta_2}{dp} + \ldots\right), \qquad (50)$$

when m is even, and

$$\tau = j\left(\frac{d\beta_r}{dp} + \frac{d\beta_1}{dp} + \frac{d\beta_1}{dp} + \frac{d\beta_2}{dp} + \ldots\right), \qquad (51)$$

Table 1. Filter Having Quasi-Symmetrical Characteristics

k	e^{2a_r} (ideal with maximum flatness)	e^{2a_r} (effective with maximum flatness)	$T(p)_r$
1	$1 + M(\omega-\omega_o)^2$	$1+H\dfrac{(\omega^2+\omega_o^2)(\omega^2-\omega_o^2)^2}{\omega^2}$	$\dfrac{p}{g^3(p)}$
2	$1 + M(\omega-\omega_o)^4$	$1+H\dfrac{(\omega^2-\omega_o^2)^4}{\omega^2}$	$\dfrac{p^2}{g^4(p)}$
3	$1 + M(\omega-\omega_o)^6$	$1+H\dfrac{(\omega^2+\omega_o^2)(\omega^2-\omega_o^2)^6}{\omega^4}$	$\dfrac{p^2}{g^7(p)}$
4	$1 + M(\omega-\omega_o)^8$	$1+H\dfrac{(\omega^2-\omega_o^2)^8}{\omega^4}$	$\dfrac{p^3}{g^8(p)}$
5	$1 + M(\omega-\omega_o)^{10}$	$1+H\dfrac{(\omega^2+\omega_o^2)(\omega^2-\omega_o^2)^{10}}{\omega^6}$	$\dfrac{p^3}{g^{11}(p)}$
6	$1 + M(\omega-\omega_o)^{12}$	$1+H\dfrac{(\omega^2-\omega_o^2)^{12}}{\omega^{16}}$	$\dfrac{p^3}{g^{12}(p)}$

when m is odd.

The conditions considered for the attenuation characteristic are also applicable for the group-delay.

Attenuation and group-delay curves are shown on the next pages in Fig. 7, 8, for the transfer functions having order k equal to 3 and 4.

III. Design Procedure

The first step in design is the choice of the order k for the desired filter. For the physically realizable functions, the two cases of even and odd are to be considered separately.

1. Even k.

On the basis of Equation (36) and the considerations of the development of the conditions for the symmtery, the basic equation can be written as

$$e^{2a_r} = 1 + e\,(e^{2a_m} - 1)\,\frac{(\omega^2 - \omega_o^2)^{2k}}{N\,\omega^k}, \tag{52}$$

assuming that the constant N has such a value that the function $(\omega^2 - \omega_o^2)^{2k}/N\omega^k$ becomes 1 for the angular frequencies ω_a and ω_b limiting the pass-band.

The following conditions are derived therefrom:

$$(\omega^2 - \omega_o^2)^{2k} = N\,\omega_a^{\,k};$$

$$(\omega_b^2 - \omega_o^2)^{2k} = N\,\omega_b^k.$$

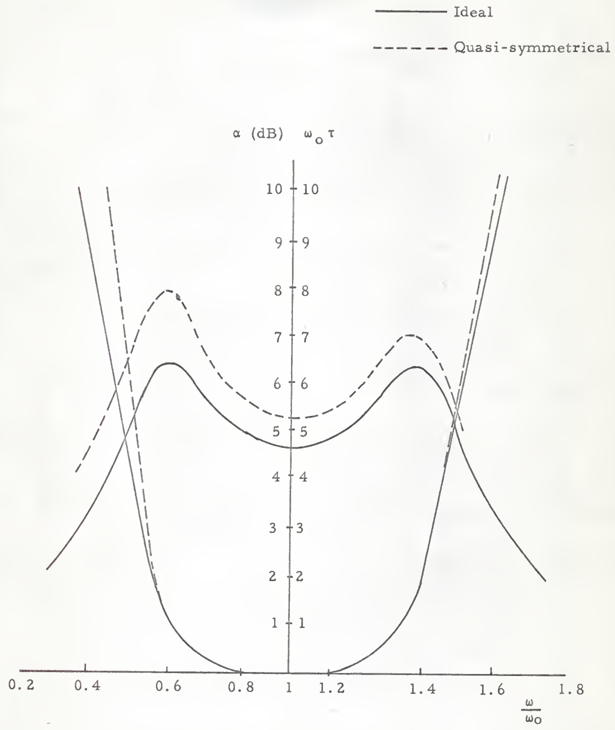

Fig. 7 Attenuation (lower graphs) and group-delay (upper graphs) characteristics for a filter having $\kappa = 3$ in the ideal case, in the real-symmetrical case.

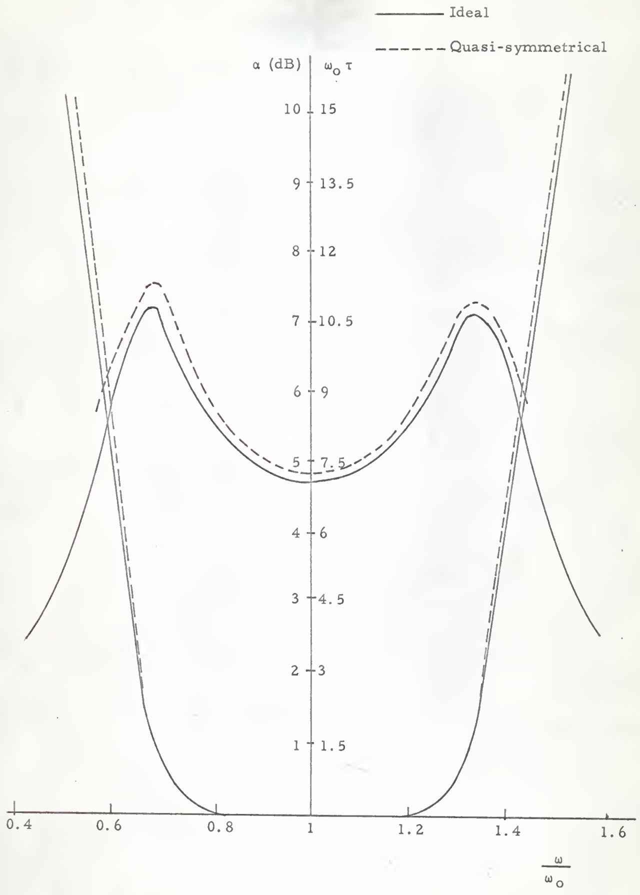

Fig. 8 Attenuation (lower graphs) and group-delay (upper graphs) characteristics for a filter having $\kappa = 4$ in the ideal case, in the real quasi-symmetrical case.

From these conditions, one can deduce that

$$\left(\frac{\omega_a^2 - \omega_o^2}{\omega_b^2 - \omega_o^2}\right)^{2k} = \left(\frac{\omega_a}{\omega_b}\right)^k \tag{53}$$

and

$$N^2 = \left(\frac{(\omega_a^2 - \omega_o^2)^2\,(\omega_b^2 - \omega_o^2)^2}{\omega_a\,\omega_b}\right)^k. \tag{54}$$

From Equation (53) we obtain:

$$\frac{\omega_a^{\,2} - \omega_o^{\,2}}{\omega_b^{\,2} - \omega_o^{\,2}} = -\sqrt{\frac{\omega_a}{\omega_b}}\,, \tag{55}$$

and now ω_o as a function of ω_a and ω_b must be deduced. After some manipulation we have:

$$\frac{\omega_o}{\omega_m} = \sqrt{\sqrt{1 - \frac{\Omega_r^{\,2}}{4}}\left(2 - 1 - \frac{\Omega_r^{\,2}}{4}\right)}. \tag{56}$$

where the center angular frequency $\omega_m = \frac{\omega_a + \omega_b}{2}$ and the relative band-width $\Omega_r = \frac{\omega_b - \omega_a}{\omega_m}$. Fig. 9 has been derived from Equation (56). Regarding the constant N, its value can be deduced from Equation (54) after introduction of the correct value of ω given by Equation (56). Equation (54) can also be simplified to

$$n = \frac{(N)^{1/k}}{\omega_m \Omega^2} = 2 \, \frac{1 - \sqrt{1 - \frac{\Omega_r^2}{4}}}{\frac{\Omega_r^2}{4}} \quad ; \tag{57}$$

This gives the values of a more convenient non-dimensional quantity n, instead of the values of N. Formula (57) is graphically shown in Fig. 10. From Fig. 9 and Fig. 10 it is thus possible to obtain the values of ω_o and N to be introduced in (52).

2. Odd k.

In the case of odd k, the following formula can be written on the basis of Equation (35) and considerations of conditions for arithmetic symmetry:

$$e^{2a_r} = 1 + (e^{2a_m} - 1) \frac{(\omega^2 + \omega_o^2)\,(\omega^2 - \omega_o^2)^{2k}}{T\,\omega^{k+1}} \tag{58}$$

Like the previous case, the constant T should be so selected that the function

$$\frac{(\omega^2 + \omega_o^2)\,(\omega^2 - \omega_o^2)^{2k}}{T\,\omega^{k+1}}$$

assumes the value 1 for the angular frequencies ω_a and ω_b limiting the pass-band.

Expressions corresponding to Equation (53), Equation (54) can thus be obtained:

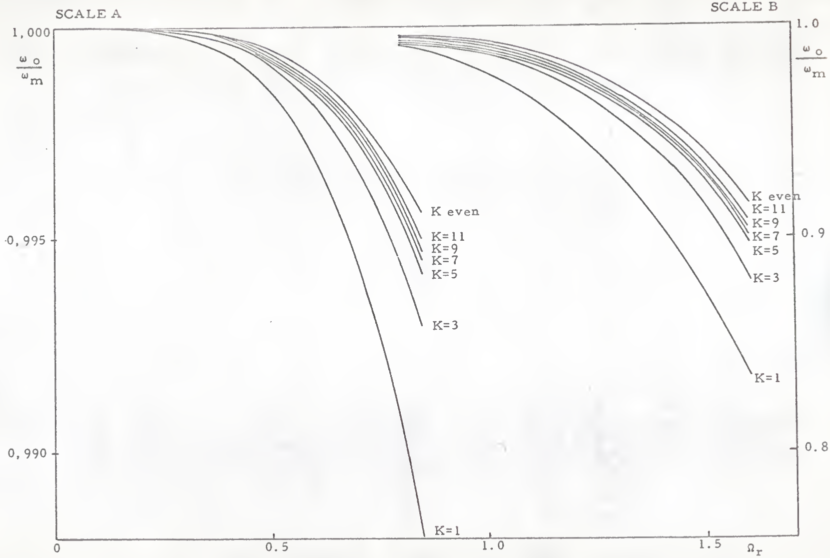

Fig. 9 Position of the minimum attenuation angular frequency ω_o with respect to the central angular frequency ω_m as a function of the relative bandwidth Ω_r, for even κ and for various values of odd κ; for odd κ approaching infinite the value of ω_o approaches the ω_c of even κ. Scales A and B are for for the curves to the left and to the right, respectively.

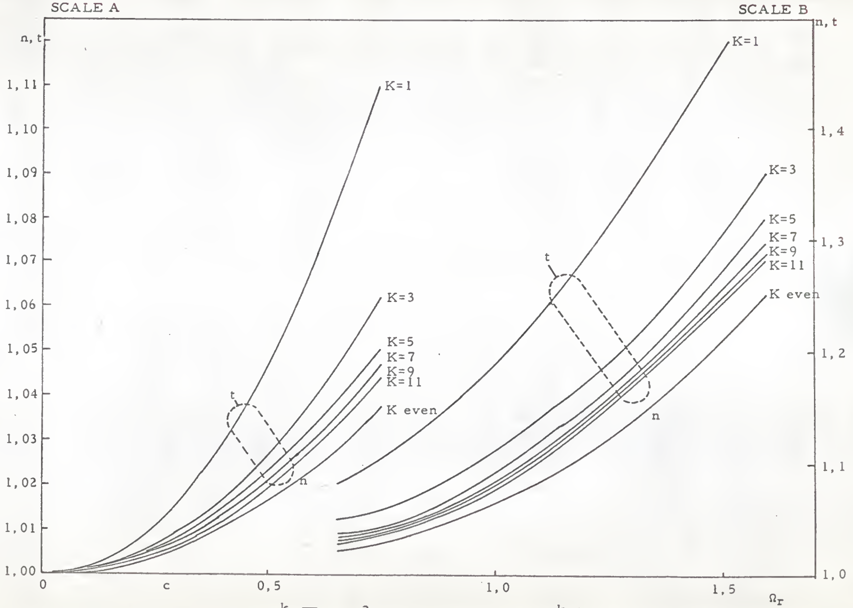

Fig. 10 Values of $n = \sqrt[k]{N/\omega_m\Omega^2}$ for even κ, and of $t = \sqrt[k]{T/2\omega_m/\omega_m\Omega^2}$ for odd κ.

$$\frac{\omega_a^2 + \omega_o^2}{\omega_b^2 + \omega_o^2} \left(\frac{\omega_a^2 - \omega_o^2}{\omega_b^2 - \omega_o^2} \right)^{2k} = \left(\frac{\omega_a}{\omega_b} \right)^{k+1}, \tag{59}$$

and

$$T^2 = \frac{(\omega_a^2 + \omega_o^2)(\omega_b^2 + \omega_o^2)}{\omega_a \, \omega_b} \left(\frac{(\omega_a^2 - \omega_o^2)^2 (\omega_b^2 - \omega_o^2)^2}{\omega_a \, \omega_b} \right)^{k} \tag{60}$$

As before, the constant T can be represented by means of a convenient non-dimensional quantity, given by the expression

$$t = \frac{\sqrt[k]{\dfrac{T}{2\,\omega_m}}}{\omega_m \Omega^2} . \tag{61}$$

Fig. 9 and Fig. 10 are the expressions of Equations of (59) and (61).

SUMMARY

Two approaches provide us with means of obtaining arithmetically symmetrical band-pass filters characteristics function of arbitrary shape.

In the Characteristic Function Approach, the symmetry is characterized by a deviation function which must be reduced by a compensating function. This involves some numerical or graphical approximation.

The method for filter attenuation has been extended to the design of band-pass filter with approximating constant group-delay.

In the Butterworth Approach, bandpass transfer functions without other zeros than at zero and infinite frequencies are analyzed according to the Butterworth bandpass behavior for obtaining the maximum possible symmetry of attenuation and group-delay characteristics.

While the Characteristic Function Approach demands a great deal of time to evaluate the deviation function and to find a compensating function, the Butterworth Criterion Approach gives a fairly straightforward technique to obtain arithmetically symmetrical characteristics.

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A STUDY OF ARITHMETICALLY SYMMETRICAL BANDPASS FILTERS

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AN ABSTRACT

The problem of designing wide bandpass filters with arithmetically symmetrical characteristics is investigated. Two approaches are considered. In the characteristic function approach it is shown that the symmetry can only be approximated by a finite, lumped element network, and a method is given to carry out this approximation. This method is given to carry out this approximation. This method consists of a periodic transformation of a suitable lowpass characteristic function, truncation of the resulting infinite product, and finally correction for the truncation error. Another approach, that of the Butterworth criterion, is also described to obtain arithmetically symmetrical characteristics. Conditions for arithmetical symmetry are analyzed.